Bursting Oscillations in Nonlinear Oscillators With Slowly Varying Excitation

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Abstract
This work represents the extension of a recently published paper by the authors, where mechanical manifestations of bursting oscillations in slowly rotating systems are presented. One of these mechanical systems consists of a mass moving along a slowly rotating frame, which creates a slowly varying external excitation. Here, the aim is to link its mathematical model with Izhikevich's methodology and classification for bursting oscillations. In this respect, the existence of a 'fold – fold' hysteresis loop of a point-point type burster is recognised and explained in detail. Thus, in the system considered, bursting oscillations occur becouse the rate of convergence between the branches of the slow flow is relatively weak. The influence of some system's parametrs on the characteristics of fast motion is illustrated.

Keywords
Bursting, oscillations, slowly varying external excitation, hysteresis

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Introduction

Bursting oscillations represent mixed-mode oscillations which occur in systems containing fast and slow variables, with the slow variable modulating the fast oscillation. Besides their appearance in a wide range of natural systems from biology [1, 2] and chemistry [3] to physics [4], they have also been recognised as beneficial in engineering systems [5].

Recent studies [6, 7] have shown that bursting oscillations can appear in the systems governed by nonlinear non-autonomous equations having the form

$$\ddot{x} + \delta\,\dot{x} + f(x) = \gamma\cos(\omega\,t), \tag{1}$$

where x is a non-dimensional variable that is henceforth referred to as the displacement; overdots denote differentiation with respect to non-dimensional time t; δ is a non-dimensional damping coefficient; γ is the magnitude of the applied harmonic excitation; ω is the angular frequency of the harmonic excitation, and this is considered to be small; the restoring force $f(x)$ corresponds to a bistable Duffing one $f(x) = -x + x^3$ in [5], while in [6], it comprises a negative or positive linear term, a negative cubic term and a positive quintic term. Further qualitative and quantitative insights into the behaviour of the former case have recently been presented in [8, 9]. The illustrations of bursting oscillations appearing in this system are shown for the sake of the reader in Figure 1.

It should be noted that a common approach in visualizing Eq. (1) was to establish the analogy with the motion of a particle in a potential well in a translatory moving cart (Figure 2a,b), whose acceleration is $a_b(t) = \gamma\cos(\omega t)$ [10]. In the case of a bistable potential well shown in Figure 2a, one could ask the question regarding the feasibility of such a motion and the transition from one potential well to another one. Or, is there a minimum value for ω, below/above which the desired response is no longer possible? The negative answer to the first question is also related to the case of a monostable potential well shown in Figure 2b. Hence, it might be more suitable to establish the analogy with the motion of a particle in a rotating potential well (Figure 2c,d), which is to be demonstrated in this paper. In this way, the bursting oscillations can be exghibited both in the bistable (Figure 2c) and monostable potential well (Figure 2d).

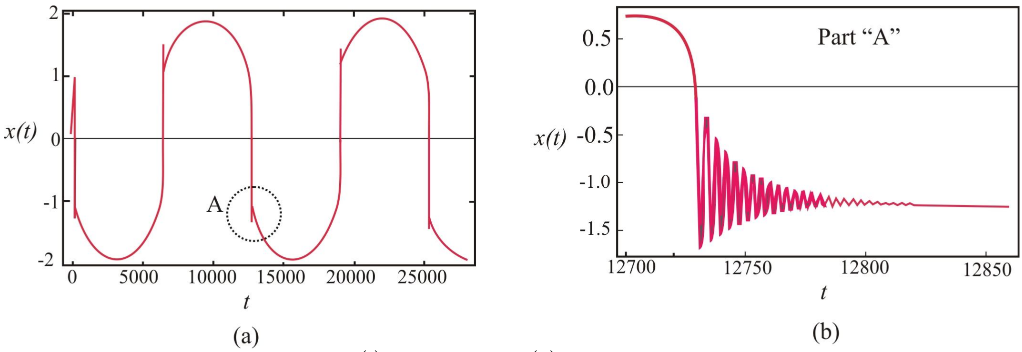

Figure 1. a) The solution $x(t)$ of Eq. (1) for $f(x)=-x+x^3$, $\delta=0.1$, $\gamma=5$ and $\omega=0.0005$; b) Enlarged part A

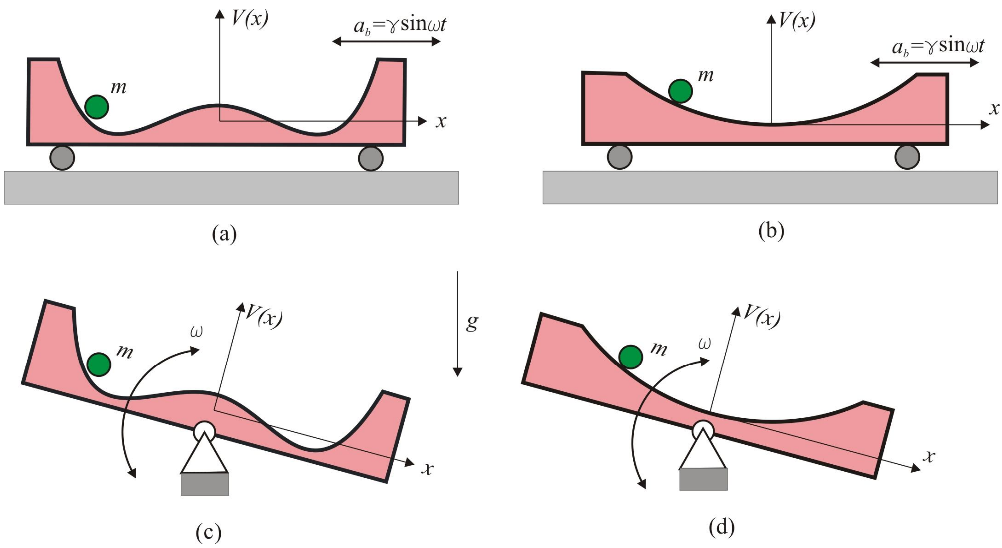

Figure 2. Analogy with the motion of a particle in a translatory and rotating potential well: a, c) Bistable potential well; b, d) Monostable potential well

Motivated by the consideration presented in [6], the authors introduced a simple mechanical system in which bursting oscillations can appear, containing a slowly rotating well [11]. Unlike the results given in [6], the design of this mechanical system enables one to create a bistable Duffing, pure cubic and hardening Duffing oscillator and to show that bursting oscillations can appear in all of them when the external excitation is slowly varying. This paper is structured as follows. Section 1 contains the explanations of the mechanical system and its design, which is followed with the derivation of the approximated equation of motion. Then, the subsequent section is concerned with the analysis of bursting oscillation, aiming to provide certain links with the existing Izhikevich's classification of bursters with respect to the hysteresis loop created. The illustration of the bursting oscillations in all three types of oscillators mentioned related to this hysteresis loop is presented in Section 4.

1. Mechanical model

A mechanical system considered is presented in Figure 3. It consists of a heavy slider moving along a slowly rotating frame. The massless frame is connected with the base via a hinge O and the frame is rotating around the axes through O, orthogonal to the frame's plane. In fact, one can also assume that the frame can swing from one side to another one between certain limited values of the angle φ. This rotation of the frame is very slow, i.e. with a low angular frequency. Such motion of the frame creates a slowly varying external excitation. The linear spring with the stiffness *k* connects the frame with the slider, and its undeformed lenght is l_0. The whole system is immersed in viscous fluid, and the corresponding dissipative force is modelled as depending linearly on the absolute velocity of the slider. The relative motion of the slider along the frame is defined by the coordinate *x* and the aim is to obtain the equation of this motion. The parameter *b* defines a possible stable equilibrium position of the slider when φ = 0.

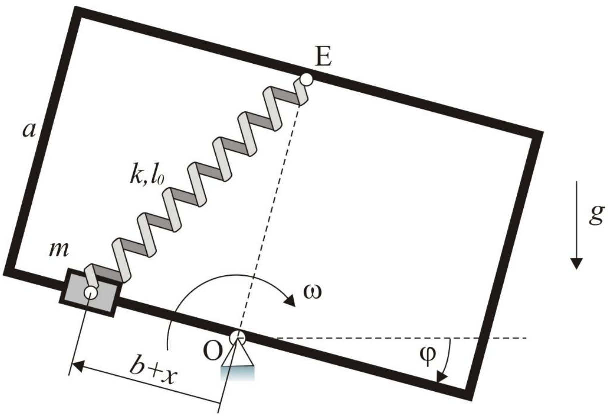

Figure 3. Mechanical model of the system under considerations

The kinetic energy of the slider is given by

$$E_k = \frac{1}{2} m \left[\dot{x}^2 + (b+x)^2 \omega^2 \right], \tag{2}$$

while the potential energy *V* has the form

$$V = m\,g\,(b+x)\sin\varphi + \frac{1}{2} k \left(\sqrt{a^2 + (b+x)^2} - l_0 \right)^2. \tag{3}$$

Taking into account Eqs. (2) and (3), Lagrange's equation of the second kind with the expansion of the restoring force into a series up to the cubic term and the assumption of negligible ω^2, can be written down as

$$\ddot{X} + \delta\,\dot{X} + r\,X + p\,X^3 = q\sin(\omega t), \tag{4}$$

where

$$X = \frac{b+x}{a},\ \delta = \frac{c}{m}, r = \frac{c(1-D)}{m},\ p = \frac{k\,D}{2m},\ \ q = -\frac{g}{a},\ D = \frac{l_0}{a}. \tag{5}$$

Equation (4) corresponds to a forced nonlinear oscillator of the Duffing type. One of the main characteristics of this equation is the existence of the slow external forcing due to the small ω. Besides this, it should be emphasized that the coefficient in front of the linear term in the restoring force *r* can have negative (D > 1), zero (D =1) or positive (D < 1) values. This implies that this equation of motion can have the form a bistable Duffing equation, a pure cubic equation or a hardening Duffing equation, which depends how l_0 is related to a (respectively, all three cases are obtained for $l_0 < a$, $l_0 = a$ or $l_0 > a$). In all these cases, for the appropriate choice of the system parameters, bursting oscillations can be achieved. This is different from the study presented in [6], where only the case of a system with a negative linear stiffness was considered as producing bursting oscillations.

2. Analysis of bursting oscillation and links with Izhikevich's methodology and classification

The system described by Eq. (4) can be represented as a system of the following first-order differential equations

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\delta x_2 - r x_1 - p x_1^3 + u, \tag{6a,b}$$

with

$$u = q \sin(\omega t). \tag{7}$$

Taking into account that the parameter ω takes small values, the variable u could be treated as slowly changing

$$\dot{u} = w = q\,\omega \cos \omega t = \omega \sqrt{q^2 - u^2}\ . \tag{8}$$

The form of Eqs. (6) and (7) is suitable for using geometric bifurcation theory and for making links with Izhickevich's methodology for analysing bursting oscillations [1].

First, it can be identified that Eqs. (6) and (7) describe the behaviour of the fast-slow burster whose general form reads as

$$\dot{x} = f(x,u),$$
$$\dot{u} = \mu\, g(x,u), \tag{9a,b}$$

where the vector $x \in \mathbf{R}^m$ describes a fast subsystem and the vector $u \in \mathbf{R}^k$ describes a slow modulation. In Eq. (9b), the parameter μ denotes a slow variable. The key feature in studying this type of burster is the method of dissection where the dynamics of x and u can be considered separately [12]. In the case considered, the vectors $x \in \mathbf{R}^2$ and $u \in \mathbf{R}^2$ can be referred to as a '2+2' burster. Due to the fact that the fast subsystem is two-dimensional, this burster is classified as planar.

The slow variable in the system considered is the angular frequency ω of the frame. It should be highlighted that the rate of change of u defined by Eq. (8) does not depend on x, i.e. the motion of the frame is not influenced by the motion of the slider. This could be seen as a certain extension of the existing burster classification. However, it seems reasonable to suppose that certain bursting oscillations in living organisms can be forced by external periodic excitations that are independent of this process, as in the case of a circadian cycle, for example.

In the system governed by Eqs. (6) and (7), bursting oscillations occur due to the existence of bifurcation of the equilibrium. Namely, due to the rotation of the frame, a stable equilibrium is being continually created and destroyed. This causes a jump of the slider to a new equilibrium position. Due to the existence of damping, this rate of convergence to the equilibrium position is slow, compared to the small parameter. The existence of bistability of the potential energy is only one of the possibility when burting oscillations can be created, but not a necessary one. In order to achieve a jump, a hysteresis loop must exist. Even in the case of a monostable potential energy, there is a possibility for creating such a loop [11]. The hysteresis loop arises from a nullcline defined by Eqs. (6) and (7), which read as

$$x_2 = 0, \tag{10}$$
$$u = r x_1 + p x_1^3, \tag{11}$$

$$u = \pm q. \tag{12}$$

It should be pointed out that the nullcline for the subsystem defined by Eq. (8) periodically changes between q and $-q$, as presented by Eq. (12). The corresponding geometric presentation is shown in Figure 4a, where the nullcline for Eq. (6b) defined by Eq. (11) is depicted by the solid *S*-shaped curve. Two vertical planes are also plotted, which are tangential to it in two points labeled by NS_1 and NS_2. This can be understood as a motion of vertical plane along u axes, labelled by the thick line with a double arrow head with the velocity defined by Eq. (8). Two vertical planes shown in Figure 5a represent limiting cases and contains the nullcline defined by Eq. (12). Besides these two points $\mathrm{NS}_1, \mathrm{NS}_2$, there are two more points of interest: N_1 and N_2. Note that 'NS' stands for 'node-saddle', while 'N' stands for 'node'. Namely, the notation NS should suggest a transition from a stable equilibrium (node) into an unstable one (saddle). This occurs when a vertical plane represents a tangent plane for the nullcline defined by Eq. (11) and when the stable node on the other branch of the nullcline exists (or it is created).

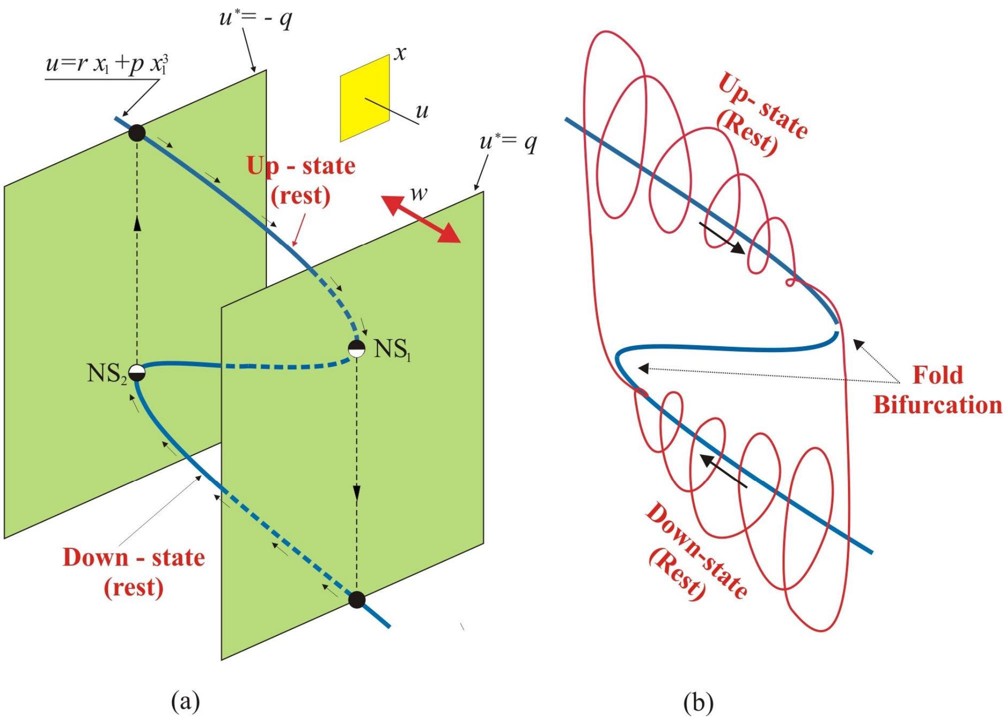

Figure 4. a) Mechanism of creating a 'point – point' hysteresis loop by slowly changing the variable u; b) 'fold – fold' bursting

Having this in mind, bursting oscillations can be represented by a three-dimensional phase trajectory around the nullcline defined by Eq. (11), as shown in Figure 4b. It can be seen that this phase trajectory jumps from one branch of the nullcline to another one in the points $\mathrm{NS}_1, \mathrm{NS}_2$, N_1 and N_2.

One of the most interesting characteristics of the observed behaviour is that bursting oscillations occur despite the fact that the fast subsystem does not have a stable limit cycle. Izhikevich [1] stressed this fact and classified this system as a 'fold – fold' hysteresis loop of 'point – point' type. The term 'fold' is due to the fact that fold bifurcation is synonyms for saddle-node bifurcation. The term 'point' should emphasize that the fast subsystem has no stable limit cycle. Note also that the parameter u represents a slowly changing parameter in the system due to the low value of ω. This parameter can be treated as a quasistatic bifurcation parameter. By changing this parameter, one can control the system behavior.

2. Illustrations of behaviour

Figure 5 illustrates the behavior corresponding to the bistable case ($r < 0$, $D > 1$). Figure 5a contains the presentations of bursting oscillations in the u-x_1 diagram obtained by numerical integration of the equation of motion (4) and plotted as the blue dashed and green dashed-dotted lines. Here, the motion is superimposed on the *S*-shaped curve. The slow variable must achieve and exceed a critical value u_{cr} labelled in Figure 5b. In this position, the unstable (S - saddle) and stable (N – node) collide and mutually annihilate. There exist only one fixed point (N) and the slider jumps towards it. The slider does not stop immediately in this equilibrium, but it oscillates around it, which is manifested as bursting oscillations. Figure 5b and 5c contain the corresponding time-history diagrams for the displacement and velocity. What should be noticed here is a very large magnitude of the velocity of the slider, as seen in Figure 5d). The parts related to bursting oscillations are labelled by I and those related to the jumps by II and III (Figure 5a and b).

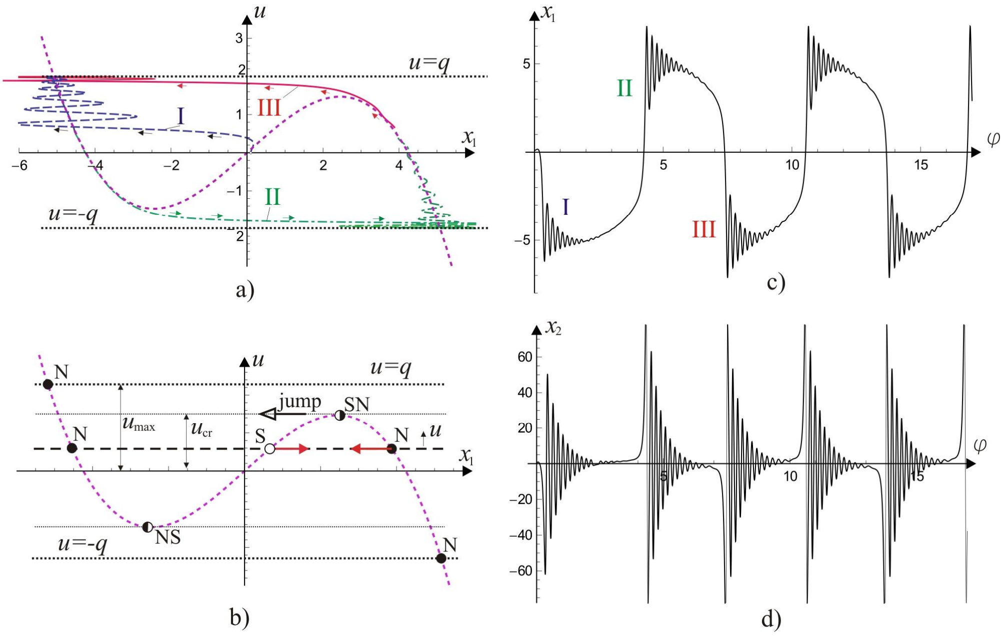

Figure 5. Bistable case: a) Diagram u versus x_1 for $\omega = 0.05$, $q = 2$, $\delta = 0.2$, $r = -0.9$ and $p = 0.05$. b) Jump and hysteresis; c) Displacement in terms of the angle φ; d) Velocity in terms of of the angle φ

The analogous presentations of the system behavior corresponding to two monostable cases are given in Figure 6 and 7. The case of the hardening Duffing oscillator is shown in Figure 6, including the numerically obtained response superimposed on the *S*-shaped curve (Figure 6a), a time-history diagram for the displacement (Figure 6b) and velocity (Figure 6c). The analogous presentations are given also for the pure cubic oscillator in Figure 7. It is apparent that the maximal velocity is lower than in the case of the bistable system.

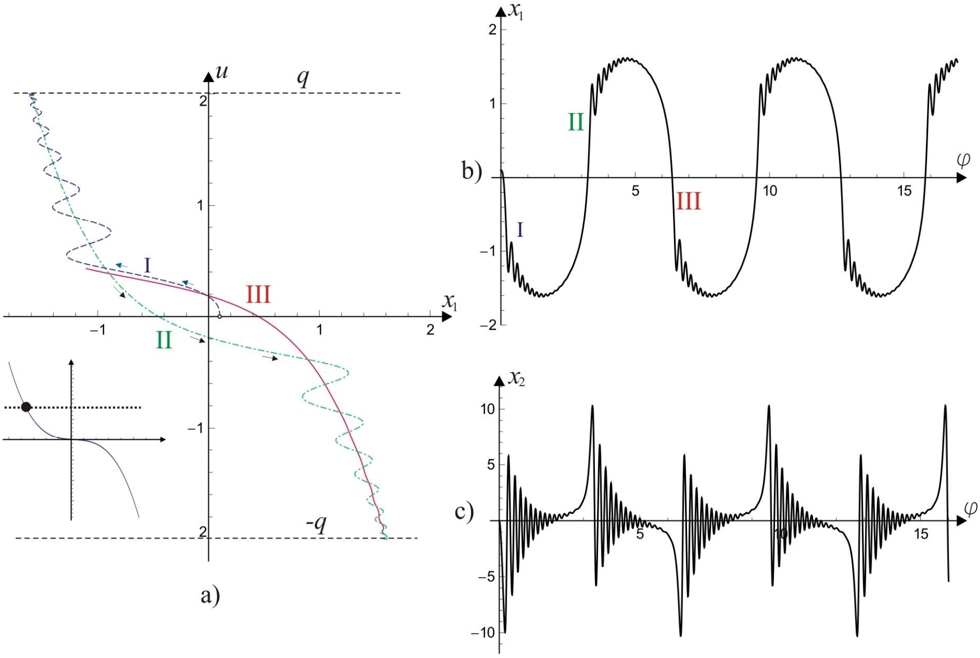

Figure 6. Monostable case - hardening Duffing oscillator: a) Diagram u versus x_1 for $\omega = 0.05$, $q = 2$, $\delta = 0.2$, $r = 0.1$ and $p = 0.45$. b) Jump and hysteresis; c) Displacement in terms of the angle φ; d) Velocity in terms of the angle φ

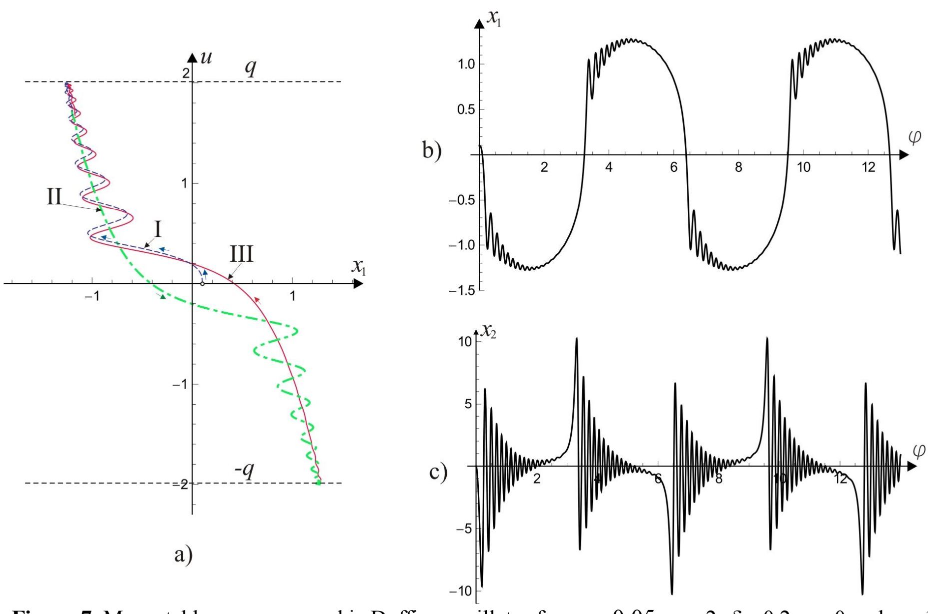

Figure 7. Monostable case - pure cubic Duffing oscillator for $\omega = 0.05$, $q = 2$, $\delta = 0.2$, $r = 0$ and $p = 1$: a) Diagram u versus x_1; b) Jump and hysteresis; c) Displacement in terms of the angle φ; d) Velocity in terms of of the angle φ

Conclusions

This paper is concerned with the occurrence of bursting oscillations in a mechanical system whose motion is described by the Duffing equation with slowly changing external excitation. The design of this mechanical system includes a slowly rotating frame, which creates slowly changing external excitation for a slider moving along it. A spring is used to attach the slider to the frame, and depending on its length in the undeformed position with respect to the height of the frame, the resulting equation of motion of the slider can be: a bistable Duffing equation, a pure cubic equation or a hardening Duffing equation. It is illustrated how bursting oscillations are created and exhibited in all these three cases. In addition, the study provides a link with Izhikevich's classification of bursting oscillations as well as the illustration of the existing hysteresis loop in certain illustration of the bursting phenomenon.

Acknowledgments

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